

The approach to vortex reconnection

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(Dated: June 16, 2010)

Abstract

We present numerical solutions of the Gross–Pitaevskii equation corresponding to reconnecting vortex lines. We determine the separation of vortices as a function of time during the approach to reconnection, and study the formation of pyramidal vortex structures. Results are compared with analytical work and numerical studies based on the vortex filament method.

PACS numbers: 67.25.dk, 47.37.+q 03.75.Kk

I. MOTIVATION

The discrete nature of vorticity in quantum fluids (superfluid ^4He , ^3He and atomic Bose–Einstein condensates) makes it possible to give physical meaning to vortex reconnections. The importance of vortex reconnections in turbulence cannot be overestimated. Reconnections randomise the velocity field thus sustaining the turbulent vortex tangle in a steady state [1] and changing the flow’s topology [2]. By triggering a Kelvin wave cascade [3–7], reconnections are responsible for the decay of turbulent kinetic energy at very low temperatures [8]. The dependence of the vortex reconnection frequency on the observed vortex line density is therefore an important quantity [9, 10].

In quantum fluids, the existence of vortex reconnections was first conjectured by Schwarz [1]; it was then theoretically demonstrated by Koplik and Levine [11], and experimentally observed at the University of Maryland [12, 13] using micron-size solid hydrogen tracers trapped in superfluid vortex lines. More precisely, the Maryland investigators inferred the existence of reconnections by the sudden movement of trapped particles. In doing so they relied on numerical calculations by de Waele and Aarts [14], who reported that the distance between two vortices, δ , varies as

$$\delta(t) = (\kappa/2\pi)^{1/2} \sqrt{t_0 - t}, \quad (1)$$

where κ is the quantum of circulation, t is time and t_0 is the time of reconnection. This $(t_0 - t)^{1/2}$ scaling agrees with the approximate analytic solution of the nonlinear Schroedinger equation (also called the Gross–Pitaevskii equation, or GPE for short) found by Nazarenko and West [15]. De Waele and Aarts also claimed that, on their way toward a reconnection, the vortices form a universal pyramidal cusp structure which is independent of the initial condition.

The numerical calculations of de Waele and Aarts [14] were based on the vortex filament model pioneered by Schwarz [1], which assumes that the vortex core is much smaller than any other length scale in the problem. The vortex filament model thus breaks down in the vicinity of a reconnection. It is therefore instructive to repeat the calculation seeking full numerical solutions of the GPE, which does not suffer such limitation and is a well-known model of superfluidity. This is indeed what we plan to do in this paper, together with revisiting the claim of universality of the reconnecting pyramidal cusp geometry.

II. MODEL

The single-particle complex wavefunction $\psi(\mathbf{r}, t)$ for N bosons of mass m obeys the 3-dimensional time dependent GPE

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi - E\psi + V_0 \psi |\psi|^2, \quad (2)$$

where \mathbf{r} is the position, V_0 is the strength of the delta function interaction between the bosons, and E is the single particle energy. The wavefunction is normalised by the condition that $\int |\psi|^2 dV = N$. If we let $\tau = \hbar/(2E)$ be the unit of time, $a = \hbar/\sqrt{2mE}$ the unit of length, and $\psi_0 = \sqrt{E/V_0}$ the unit of ψ , we obtain the following dimensionless GPE

$$-2i \frac{\partial \psi}{\partial t} = \nabla^2 \psi + (1 - |\psi|^2) \psi. \quad (3)$$

We solve Eq. 3 in an (x, y, z) periodic box, using fourth order centred differences and Runge–Kutta–Fehlberg adaptive time stepping. The typical numerical resolution is $100 \times 300 \times 120$ for example for the first calculation described in Sec. III. More details of the numerical technique and the method used to set up initial conditions will be described elsewhere [16].

III. RING–RING RECONNECTIONS

In a first set of numerical experiments, we consider the same planar vortex rings configuration of de Waele and Aarts [14]. Two vortex rings of (dimensionless) radius $R = 30$ are located on the y, z plane, initially separated by the (dimensionless) distance $\delta(0) = 20$, and moving in the positive x direction. Fig. 1 shows that, during the evolution, the rings turn into each other. The pyramidal cusp which forms (and which eventually will lead to a vortex reconnection) trails behind the rest of the moving vortex configuration.

By symmetry, the reconnection is on the $z = 0$ plane, which facilitates the computation of the minimum distance between the vortex rings, $\delta(t)$, as a function of time t , until, at $t = t_0$, the rings reconnect. We repeat the calculation for different initial separations $\delta(0) = 30, 40, 50$ and 60 . Fig. 2 shows a log-log plot of $\delta(t)$ vs t and confirms that indeed $\delta(t) \sim (t_0 - t)^{1/2}$, in agreement with Eq. 1.

We fit our $\delta(t)$ vs t data to the form

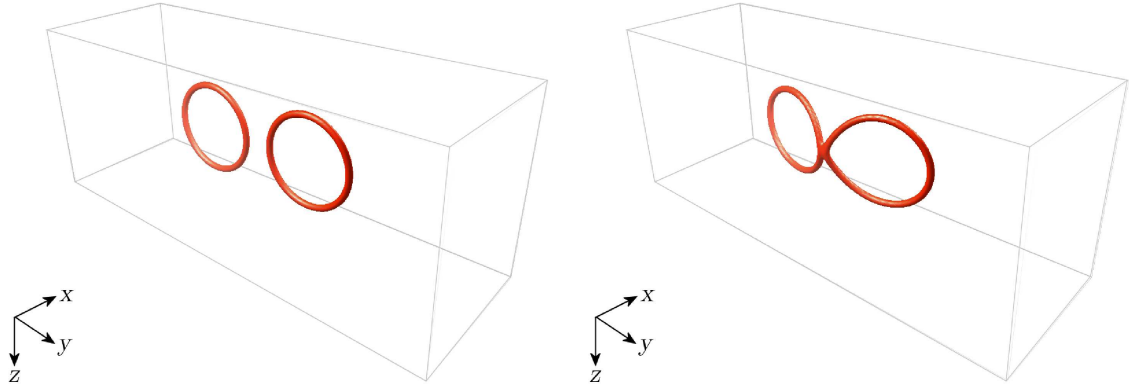


FIG. 1: (Colour online). Ring-ring reconnection. Isosurfaces of $|\psi|^2 = 0.7$ (dimensionless units) showing two rings moving along the positive x direction at (dimensionless) times $t = 0$ (left) and $t = 81$ right. Note the formation of the reconnecting cusp.

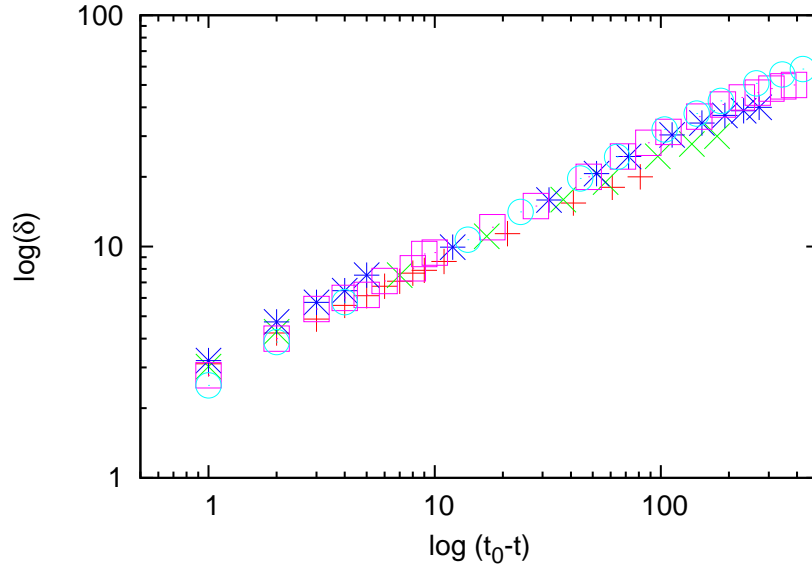


FIG. 2: (Colour online). Reconnections of vortex rings. Log-log plot of dimensionless distance $\delta(t)$ between the rings vs dimensionless time t , for various initial distances $\delta(0) = 20$ (red crosses), 30 (green diagonal crosses), 40 (blue asterisks), 50 (violet squares) and 60 (pale blue circles).

$$\delta(t) = A\sqrt{(t_0 - t)}[1 + c(t_0 - t)], \quad (4)$$

where A and c are parameters. We find that $A \gg c$, and that A increases and c decreases for increasing $\delta(0)$. More precisely we obtain $A = 2.66, 2.78, 3.18, 3.43$ and 3.63 , and

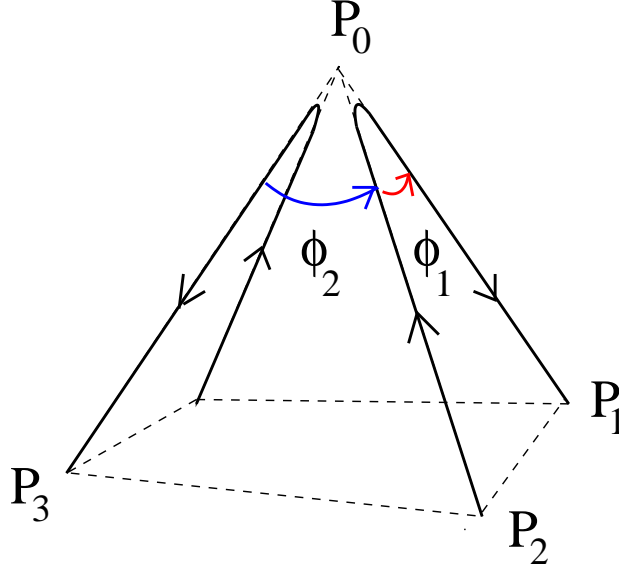


FIG. 3: (Colour online). Pyramidal reconnection cusp and definition of the angle ϕ_1 (which a vortex makes with itself) and the angle ϕ_2 (which a vortex makes with the other vortex).

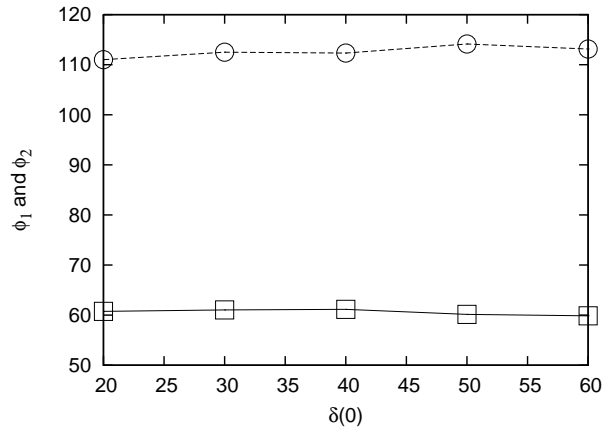


FIG. 4: Reconnections of vortex rings. Reconnecting angles ϕ_1 (top, circles) and ϕ_2 (bottom, squares) vs initial distance $\delta(0)$. The lines are to guide the eye.

$c = 2 \times 10^{-3}$, 1.06×10^{-3} , 0.87×10^{-3} , 0.67×10^{-3} and 0.52×10^{-3} , respectively for $\delta(0) = 20$, 30, 40, 50 and 60. The errors are approximately ± 0.08 for A and $\pm 0.3 \times 10^{-3}$ for c , and arise mainly from the inaccuracy in localising the axes of the vortex lines.

The Maryland group reported their results in the (dimensional) form [13]

$$\delta_d(t_d) = A_M \sqrt{\kappa_d(t_{d0} - t_d)} [1 + c_M(t_{d0} - t_d)], \quad (5)$$

(where we introduced the subscript “d” to stress that a quantity is dimensional), where

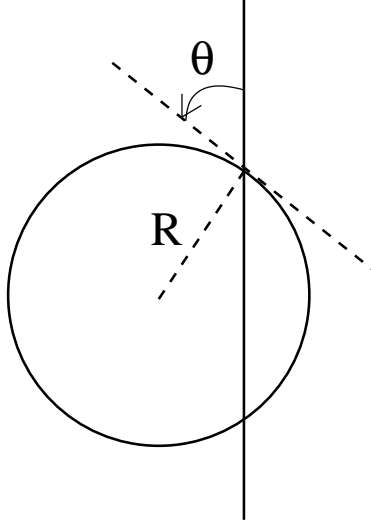


FIG. 5: Initial vortex ring–line configuration to define the angle θ .

$A_M \approx 1.2$ and $c_M \approx 0.5 \text{ s}^{-1}$ are their fitting parameters. The numerical results of de Waele and Aarts [14] correspond to $A_M = 1/\sqrt{2\pi} \approx 0.4$.

Since dimensional and dimensionless length, time and circulation are related by $\delta = \delta_d/a$, $t = t_d/\tau$ and $\kappa = \kappa_d/(a^2/\tau)$ (where $\kappa_d = 10^{-3} \text{ cm}^2/\text{s}$ and $\kappa = 2\pi$ are respectively the dimensional and dimensionless circulations), our (dimensionless) results agree with Maryland’s (dimensional) results if $A = A_M \sqrt{\tau \kappa_d/a^2} = A_M \sqrt{2\pi} \approx 3$ and $c = \tau c_M \approx 10^{-13} \approx 0$, which is indeed the case (the value of E needed to get τ to compare c with c_M can be estimated from the definition of the healing length a and the fact that $a \approx 10^{-8} \text{ cm}$).

Following de Waele and Aarts [14], we define the intravortex angle ϕ_1 and the intervortex angle ϕ_2 of the pyramidal cusp formed by the two vortices as they approach the reconnection. These angles are computed from the dot products of the vector \mathbf{r}_2 with the vectors \mathbf{r}_1 and \mathbf{r}_3 , where $\mathbf{r}_1 = \mathbf{P}_0 - \mathbf{P}_1$, $\mathbf{r}_2 = \mathbf{P}_0 - \mathbf{P}_2$, and $\mathbf{r}_3 = \mathbf{P}_0 - \mathbf{P}_3$ as shown in Fig. 3. If the total angle $\phi_{tot} = 2(\phi_1 + \phi_2)$ is equal to 360° , then the reconnection is “flat.” If $\phi_{tot} < 360^\circ$ the vortices form a pyramid.

Fig. 4 shows that reconnecting angles are approximately constant, $\phi_1 \approx 112^\circ$ and $\phi_2 \approx 61^\circ$, independently of the initial distance between the vortex rings, again in agreement with the finding of de Waele and Aarts [14], although their numerical values are slightly different: their ϕ_1 increases from 115° to 135° over a change of two orders of magnitude of the initial distance. whereas $\phi_2 \approx 25^\circ$.

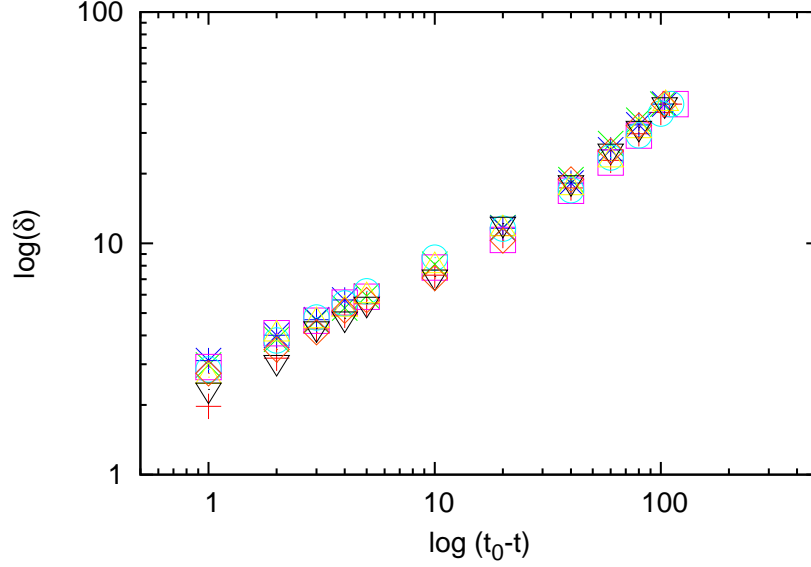


FIG. 6: (Colour online). Ring–line reconnections. Log-log plot of (dimensionless) distance $\delta(t)$ between the rings vs (dimensionless) time t , for initial angles $\theta = 60^\circ$ (red crosses), 75° (green diagonal crosses), 90° (blue asterixes), 105° (violet squares), 120° (light blue circles), 135° (yellow pointing up triangles), 150° (brown pointing down triangles) and 160° (orange diamonds).

IV. RING–LINE RECONNECTIONS

In a second set of numerical experiments, we consider the interaction of a vortex line (set in the middle of the computational box and aligned in the z direction) with a vortex ring (located in the y, z plane, with centre at $y = D$) which moves in the positive x direction). The geometry of the initial configuration can be parametrised by the angle θ between the vortex ring and the vortex line, see Fig. 5; θ ranges from $\theta = 0^\circ$ (the vortex ring passes to the left of the line) to $\theta = 180^\circ$ (the vortex ring passes to the right of the line).

Fig. 6 shows that, for ring–line reconnections, the distance between the vortices is very similar to what is determined for ring–ring reconnections. Fitting as in Eq. 4 we have $A = 2.15, 2.28, 2.28, 2.02, 2.05, 2.01, 2.27$ and 2.25 , and $c = -7.2 \times 10^{-3}, -7.2 \times 10^{-3}, -6.9 \times 10^{-3}, -7.2 \times 10^{-3}, -7.6 \times 10^{-3}, -8.9 \times 10^{-3}, -7.0 \times 10^{-3}$ and -7.2×10^{-3} respectively for initial angles $\theta = 60^\circ, 75^\circ, 90^\circ, 105^\circ, 120^\circ, 135^\circ, 150^\circ$ and 160° .

In contrast to what happens in ring–ring reconnections, there is less evidence for a universal pyramidal cusp. Fig 7 shows that ϕ_1 and ϕ_2 depend on θ , that is to say, on the initial geometry. Fig 8 shows that the total angle ϕ_{tot} (which would be 360° for a “flat”

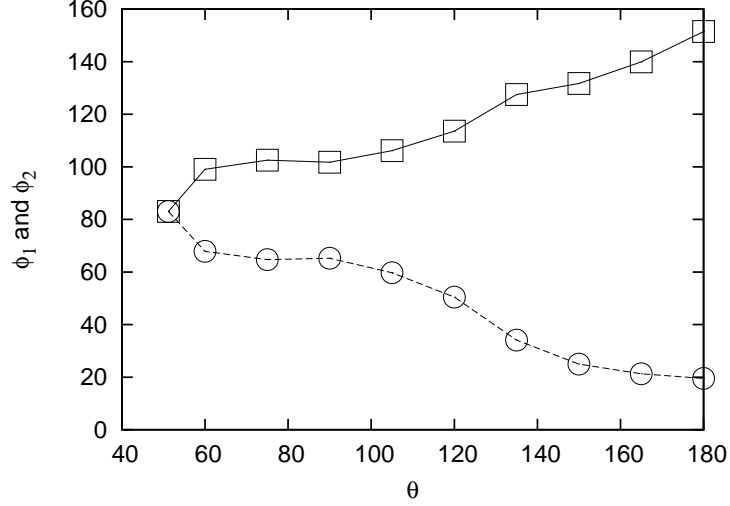


FIG. 7: Ring–line reconnections. Reconnecting angles ϕ_1 (top, squares) and ϕ_2 (bottom, circles) vs initial angle θ (all angles in degrees). The lines are to guide the eye.

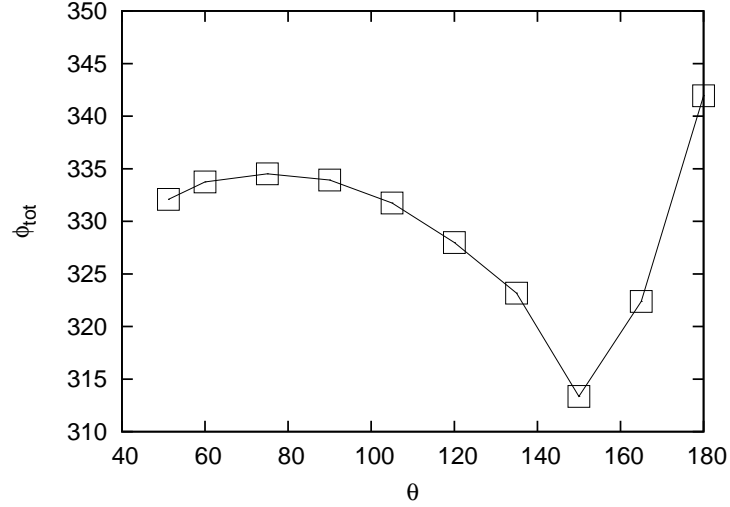


FIG. 8: Ring–line reconnections. Total cusp angle ϕ_{tot} vs initial angle θ (all angles in degrees). The line is to guide the eye.

reconnection), is not constant.

It is interesting to note that, if $\theta < 50^\circ$, when the vortex ring is sufficiently close to the vortex line, the part of the vortex ring which is near the vortex and the part of the vortex line which is near the ring are almost parallel; since they have the same circulation, they tend to rotate about each other. This motion perturbs and deflects the vortex ring, which passes past the (perturbed) vortex line without any reconnection. Because of this curious effect, shown in Fig. 9, we have no data for $\theta < 50^\circ$ in Figs. 7 and 8.

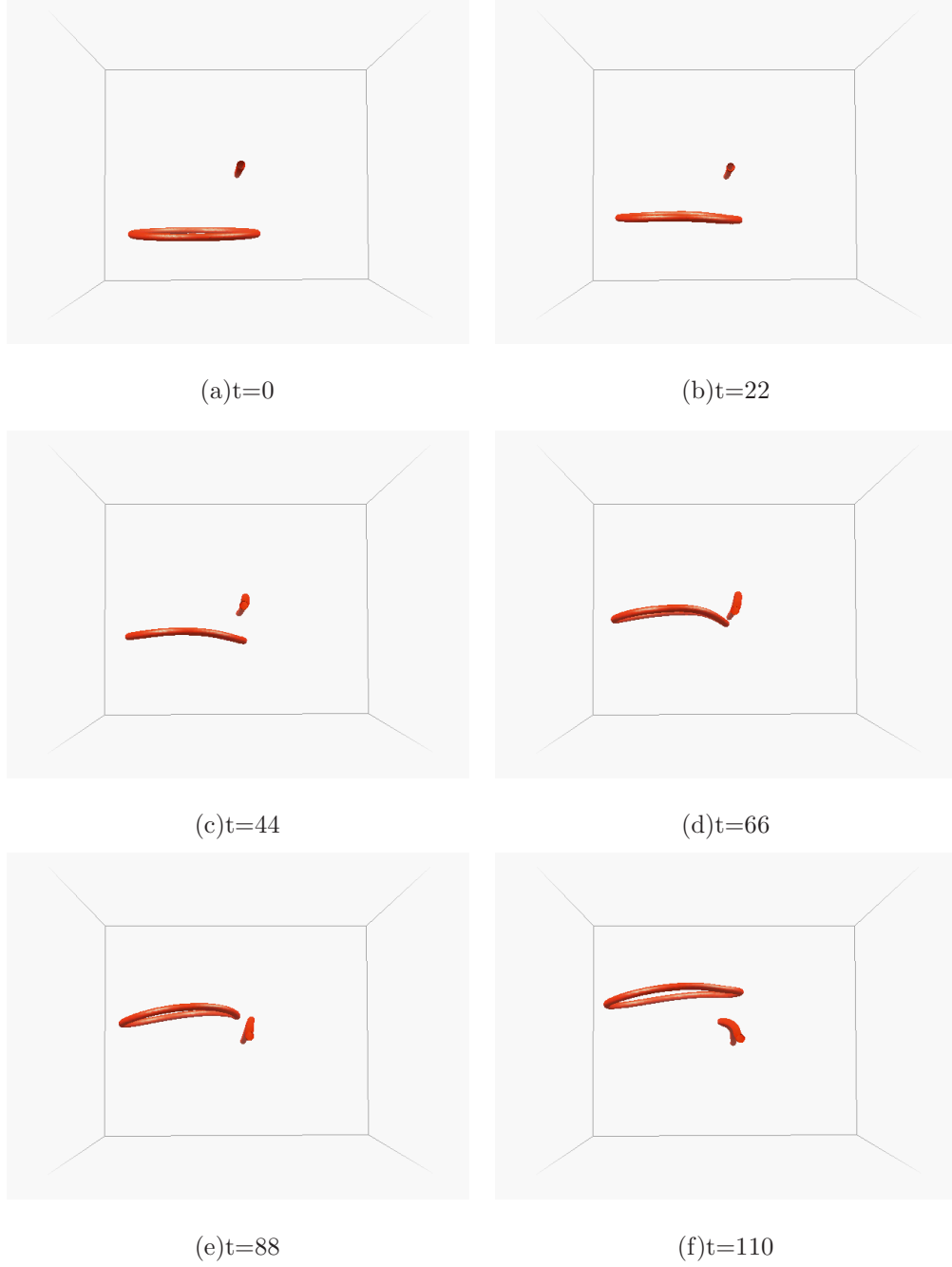


FIG. 9: (Colour online). Ring-line reconnection for $\theta = 45^\circ$, seen from the bottom, looking up the z axis. Isosurfaces of $|\psi|^2 = 0.7$ (dimensionless units) at (dimensionless) times $t = 50, 70, 90$ and 110 . The vortex ring (in the y, z plane) moves in the positive x direction past the vortex line (aligned along z) without reconnecting.

V. CONCLUSIONS

By numerically finding nonlinear solutions of the governing GPE, we have confirmed that, as vortex filaments approach a reconnection, their distance scales as $\delta \sim (t_0 - t)^{1/2}$, as predicted by Nazarenko and West [15]. By fitting our δ vs t data to Eq. 4, we find better quantitative agreement with Maryland's observations [13] than obtained with the vortex filament method [14].

Finally, we find that reconnections involve the formation of a pyramidal cusp. However, on the length scales which we can explore using the GPE (which are different from length scales described by the vortex filament method), the angles of this cusp are not universal as previously claimed [14].

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